§2.3. Coordinate Changes.

Lem: poly. map is continuous under Zariski top.

Pf: $T = (T_1, \cdots, T_m) : \mathbb{A}^n \to \mathbb{A}^m$ $\quad F^T := \tilde{T}(F) = F(T_1, \cdots, T_m)$.

$I^T := \langle F^T \mid F \in I \rangle \triangleleft k[X_1, \cdots, X_n] \quad (\forall I = I(V) \triangleleft k[X_1, \cdots, X_m])$

$V^T := V(I^T) = T^{-1}(V)$

Fact: $\left.\begin{array}{l} V = \text{hypersurface in } \mathbb{A}^m \\ F^T \neq \text{constant} \end{array}\right\} \Rightarrow V^T = \text{hypersurface in } \mathbb{A}^n$

Def An *affine change of coordinates* on $\mathbb{A}^n$ is a poly. map

$$T = (T_1, \cdots, T_n) : \mathbb{A}^n \to \mathbb{A}^n$$

s.t.
1) $\deg T_i = 1$
2) $T = \text{bijection}$.

deg 1 map = translation ∘ linear map

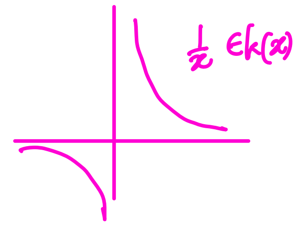
e.g. $T_i = \sum a_{ij} X_j + a_{i0}$

$\Rightarrow T = T'' \circ T' \quad (T_i' = \sum a_{ij} X_j \ , \ T_i'' = X_i + a_{i0}.)$

$T = \text{bijection} \Leftrightarrow T' = \text{bijection}$

Fact: the set of affine change of coordinates forms a gp.

§2.4 Rational functions and local rings.



$V \subseteq \mathbb{A}^n$ variety. $\Gamma(V)$ = coordinate ring.

Def: · the quotient field of $\Gamma(V)$ is called the field of rational functions on V, is written $k(V)$.

· An element in $k(V)$ is called a rational function on V.

· $\forall f \in k(V)$, $\forall P \in V$, we say that f is defined at P if $\exists\, a, b \in \Gamma(V)$ s.t. $f = a/b$ and $b(P) \neq 0$.

Rmk: 1) $f = a/b$ representation is NOT unique!

2) UFD $\Rightarrow$ essentially unique.

Example: $V = V(XW - YZ) \subseteq \mathbb{A}^4(k)$, $\Gamma(V) = k[X, Y, Z, W]/(XW - YZ)$

$f = \overline{X}/\overline{Y} = \overline{Z}/\overline{W} \in k(V)$. Then

f is defined at $P = (x, y, z, w) \Leftrightarrow y \neq 0$ or $w \neq 0$.

$\mathcal{O}_P(V) := \{ f \in k(V) \mid f \text{ defined at } P \}$ local ring of V at P.

· subring of $k(V)$

· containing $\Gamma(V)$.

$k \subset \Gamma(V) \subset \mathcal{O}_P(V) \subset k(V)$.

pole set of f $:= \{P \in V \mid f$ is NOT defined at $P\}$.

Prp. (1) pole set is an algebraic subset (of V)

(2). $\Gamma(V) = \bigcap_{P \in V} \mathcal{O}_P(V)$

Pf: $V \subset \mathbb{A}^n$, $G \in k[x_1 .. x_n]$, $\bar{G} = G \bmod I(V) \in \Gamma(V)$. $f \in k(V)$.

$J_f := \{ G \in k[x_1 \cdots x_n] \mid \bar{G} f \in \Gamma(V) \}$

- $J_f \triangleleft k[x_1, \cdots, x_n]$ ✓
- $I(V) \subseteq J_f$ ($\Rightarrow V(J_f) \subset V$) ✓
- $V(J_f) =$ pole set of f $\Rightarrow$ (1)

$$\left(\begin{aligned} \text{pole set of } f &= \bigcap_{f=\frac{a}{b}} \{P \in V \mid b(P) = 0\} = \bigcap_{G : \bar{G}f \in \Gamma(V)} \{P \in V \mid \bar{G}(P) = 0\} \\ &= \bigcap_{G \in J_f} \left(V(G) \cap V \right) = V(J_f) \cap V = V(J_f). \end{aligned} \right)$$

$\forall f \in \bigcap_{P \in V} \mathcal{O}_P(V) \Rightarrow V(J_f) = \phi$

$\Rightarrow 1 \in J_f \Rightarrow f \in \Gamma(V) \Rightarrow (2)$.

$\forall f \in \mathcal{O}_P(V)$. with $f = \frac{a}{b}$ $(b(P) \neq 0)$.

value of f at P: $f(P) := \dfrac{a(P)}{b(P)}$ (independent of a, b)

maximal ideal of V at P $\quad \mathfrak{m}_P(V) := \{ f \in \mathcal{O}_P(V) \mid f(P) = 0 \}$

$$0 \to \mathfrak{m}_P(V) \to \mathcal{O}_P(V) \underset{f \mapsto f(P)}{\xrightarrow{\text{eval}}} k \to 0$$

- $\mathcal{O}_P(V) / \mathfrak{m}_P(V) \cong k$
- $f \in \mathcal{O}_P(V)^{\times} \Leftrightarrow f(P) \neq 0$

Def: A ring is called *local*, if it has a unique maximal ideal.

Fact: R = local $\Leftrightarrow$ $R\setminus R^\times \trianglelefteq R$.

Example: $\mathcal{O}_p(V)$ is a local ring with maximal ideal $m_p(V)$.

all local property (that depend only on a neighborhood of p) are reflected in the ring $\mathcal{O}_p(V)$.

Prop: $\mathcal{O}_p(V)$ = noeth + local + domain.

交换代数 R = noeth $\Rightarrow S^{-1}R$ = noeth.

pf: $\forall\ I \trianglelefteq \mathcal{O}_p(V)$. $\Rightarrow$ $I \cap \Gamma(V) \trianglelefteq \Gamma(V)$ is f.g.

assume $I \cap \Gamma(V) = \langle f_1, \cdots, f_r \rangle$ in $\Gamma(V)$. Then

$I = \langle f_1, \cdots, f_r \rangle$ in $\mathcal{O}_p(V)$.

$$\left(\begin{aligned} \forall f \in I \ &\Rightarrow \exists b \in \Gamma(V) \text{ s.t. } b(p) \neq 0 \ \&\ bf \in \Gamma(V) \\ &\Rightarrow bf \in \Gamma(V) \cap I \Rightarrow bf = \sum a_i f_i \\ &\Rightarrow f = \sum (a_i/b) \cdot f_i \end{aligned}\right)$$

$\square$